Influence of grazing flow and dissipation effects on the acoustic boundary conditions at a lined wall

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The problem of sound propagation near a lined wall taking into account mean shear flow effects and viscous and thermal dissipation is investigated. The method of composite expansion is used to separate the inviscid part, in the core of the flow, from the boundary layer part, near the wall. Two diffusion equations for the shear stress and the heat flux are obtained in the boundary layer. The matching of the solutions of these equations with the inviscid part leads to a modified specific acoustic admittance in the core flow. Depending on the ratio of the acoustic and stationary boundary layer thicknesses, the kinematic wall condition changes gradually from continuity of normal acoustic displacement to continuity of normal acoustic mass velocity. This wall condition can be applied in dissipative silencers and in aircraft engine-duct systems. © *2001 Acoustical Society of America.* [DOI: 10.1121/1.1331678]

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I. INTRODUCTION

In this paper, the problem of acoustic propagation in a duct with parallel shear flow and transverse temperature gradient is investigated. The aim is to take into account viscothermal effects in the boundary layer near a lined wall. These effects are accounted for by modifying the boundary condition at the duct wall. The effects of temperature and velocity gradients as well as thoses caused by dissipation are concentrated in a thin layer near the wall. In the core of the flow, the fluid is considered to be an ideal gas and the velocity and the temperature vary only slowly. Finally, a new boundary condition on the wall is obtained for the sound in the core flow.

Several authors[1–8] have addressed the problem of propagation in lined flow ducts for adiabatic, inviscid sound propagation. In these cases, they have assumed continuity of displacement at the wall since it seems to be the more appropriate.[9] Nayfeh[10] has studied how viscothermal effects affect the impedance for the case where the acoustic boundary layer is much thinner than the mean flow boundary layer. In this paper, by taking into account both viscothermal effects near the wall and the effects of large stationary velocity and temperature gradients, it is shown that the effective boundary condition can be continuity of normal acoustic displacement, or continuity of normal acoustic velocity, or a mixed condition depending on the different length scales (acoustic and mean flow boundary layer thicknesses). This wall condition can be applied in lined flow ducts such as dissipative silencers and aircraft engine-duct systems.

The equations of lossy fluid mechanics linearized about a mean state are presented in Sec. II. These equations are simplified by making classical boundary layer assumptions for a flow near a plane wall and scaled to obtain dimensionless equations. An asymptotic representation of these equations is then given in Sec. III. In the core of the flow, the problem reduces to solving the equations of lossless acoustics with mean parallel shear flow and mean transverse temperature gradient. The effects of the boundary layer can be taken into account by a modified admittance of the wall. This modified admittance is found in Sec. IV by solving two diffusion equations (for the shear stress and the heat flux) in the boundary layer. In Sec. V the analysis is extended to the case of rough lined walls.

II. GENERAL EQUATIONS

The situation for the sound propagation being investigated is shown in Fig. 1, where the lined wall lies in the plane $y^*=0$ of a coordinate system (x^*,y^*,z^*). The general equations governing the linear oscillations of a gas with a mean flow are

$$\frac{\partial\rho^*}{\partial t^*}+\rho_0^*\nabla\cdot\mathbf{v}^*+\mathbf{v}_0^*\cdot\nabla\rho^*+\mathbf{v}^*\cdot\nabla\rho_0^*+\rho^*\nabla\cdot\mathbf{v}_0^*=0, \tag{1a}$$

$$\frac{\partial v^*}{\partial t^*}+(\mathbf{v}_0^*\cdot\nabla)\mathbf{v}^*+(\mathbf{v}^*\cdot\nabla)\mathbf{v}_0^*+\frac{1}{\rho_0^*}\nabla p^*+(\mathbf{v}_0^*\cdot\nabla)\mathbf{v}_0^*\frac{\rho^*}{\rho_0^*}=\frac{\mathbf{B}^*}{\rho_0^*}, \tag{1b}$$

$$\frac{\partial s^*}{\partial t^*}+\mathbf{v}_0^*\cdot\nabla s^*+\mathbf{v}^*\cdot\nabla s_0^*=\frac{Q^*}{\rho_0^*T_0^*}, \tag{1c}$$

$$p^*=c_0^{*2}\rho^*+h_0^*s^*, \tag{1d}$$

where the terms without subscript refer to fluctuating components and the subscript 0 refers to mean values $\mathbf{v}^*$ is the velocity, p^* is the pressure, ρ^* is the density, s^* is the entropy. T^* is the temperature, c_p^* is the specific heat of the gas at constant pressure, c_0^* is the adiabatic sound speed

$$c_0^{*2}(y^*)=\left(\frac{\partial p_0^*}{\partial\rho_0^*}\right)_s, \tag{2a}$$

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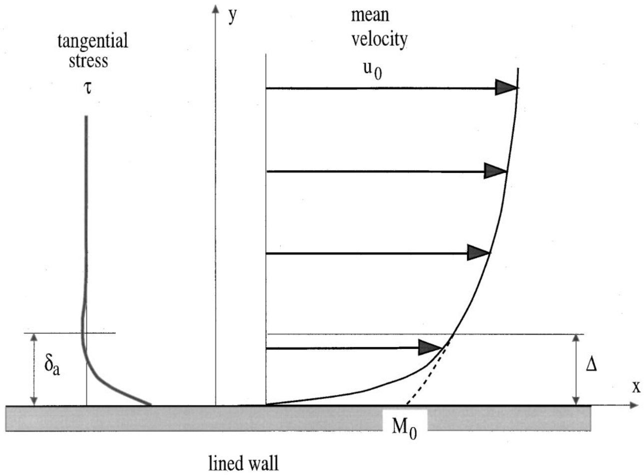

FIG. 1. Schematic description of the geometry.

and, for an ideal gas

$$h_0^*(y^*)=\left(\frac{\partial p_0^*}{\partial s_0^*}\right)_{\rho^*}=\frac{\rho_0^* c_0^{*2}}{c_p^*}. \tag{2b}$$

The sources of forces $\mathbf{B}^*$ and of heat Q^* include all the dissipation terms, and expressions for these are given in the following.

The x axis is chosen to be aligned with the mean velocity. Gradients of v_0^*, ρ_0^*, s_0^*, and p_0^* in the x and z directions are considered negligible in comparison with their gradients in the y direction normal to the wall (assuming a fully developed stationary flow). Furthermore, the stationary pressure is assumed constant in the y direction ($dp_0^*/dy^*=0$). This last assumption leads to

$$\frac{ds_0^*}{dy^*}=-\frac{c_p^*}{\rho_0^*}\frac{d\rho_0^*}{dy^*}=\frac{c_p^*}{T_0^*}\frac{dT_0^*}{dy^*}, \tag{3}$$

and to $d(\rho_0^*/c_0^{*2})/dy^*=0$.

For simplicity the sound is assumed to propagate only in the x direction (an extension to propagation in both x and z directions is straightforward). Then, by introducing dimensionless quantities, the variables may be written as follows:

$$x^*=xc_a/\omega^*,\quad y^*=yc_a/\omega^*,\quad t^*=t/\omega^*,\quad c_0^*=c_a c_0(y),$$

$$\begin{aligned} &u_0^*=c_a u_0(y),\quad u^*=c_a u(y)E,\\ &v_0^*=0,\quad v^*=c_a v(y)E,\\ &p_0^*=\rho_a c_a^2 p_0,\quad p^*=\rho_a c_a^2 p(y)E,\\ &T_0^*=T_a T_0(y),\quad T^*=T_a T(y)E,\\ &\rho_0^*=\rho_a\rho_0(y),\quad \rho^*=\rho_a\rho(y)E,\\ &s_0^*=c_p^* s_0(y),\quad s^*=c_p^* s(y)E, \end{aligned} \tag{4}$$

where the subscript a refers to dimensional properties in the core of the flow (for instance in the midline), ω^* is the frequency, c_a is the sound speed ($c_a^2=c_p^*(\gamma-1)T_a$), u and v are the velocity components in the x and y directions, and $E=\exp(-i\omega^*t^*+ik^*x^*)=\exp(-it+ikx)$ where $k=k^*c_a/\omega^*$ is the wave number.

Using expressions (4) with the above assumptions, Eqs. (1) are transformed to

$$-i\Omega\rho+\rho_0\left(iku-i\Omega\frac{d\xi}{dy}+ik\frac{du_0}{dy}\xi\right)-i\Omega\frac{d\rho_0}{dy}\xi=0, \tag{5a}$$

$$-i\Omega u-i\Omega\frac{du_0}{dy}\xi+\frac{ikp}{\rho_0}=\frac{\delta_a^2}{2}\frac{\rho_w}{\rho_0}B_x, \tag{5b}$$

$$-i\Omega v+\frac{1}{\rho_0}\frac{dp}{dy}=\frac{\delta_a^2}{2}\frac{\rho_w}{\rho_0}B_y, \tag{5c}$$

$$-i\Omega s+i\Omega\frac{1}{\rho_0}\frac{d\rho_0}{dy}\xi=\frac{\delta_a^2}{2}\frac{\rho_w}{\rho_0}\frac{Q}{T_0}, \tag{5d}$$

$$p=c_0^2\rho+s, \tag{5e}$$

where $\Omega=1-ku_0$ and $\xi=-v/i\Omega$ is the acoustical displacement in the y direction, $\delta_a=(2\mu_a\omega^*/\rho_w^*c_a^2)^{1/2}$ is the dimensionless acoustic boundary layer thickness, $\rho_w^*=\rho_a\rho_w$ is the density at the wall, and μ_a is the dynamic viscosity of the fluid. For simplicity the dynamic viscosity and the thermal conductivity are assumed to be constant and the bulk viscosity is assumed to be equal to zero; then, the dissipation terms B_x, B_y and Q are given by

$$B_x=\frac{d^2u}{dy^2}+\frac{1}{3}ik\frac{dv}{dy}+\frac{4}{3}k^2u, \tag{6a}$$

$$B_y=\frac{4}{3}\frac{d^2v}{dy^2}+\frac{1}{3}ik\frac{du}{dy}-k^2v, \tag{6b}$$

$$Q=\frac{1}{\sigma^2}\left(\frac{d^2T}{dy^2}-k^2T\right)+2(\gamma-1)\frac{du_0}{dy}\left(\frac{du}{dy}+ikv\right), \tag{6c}$$

where $\sigma^2=c_p^*\mu_a/\kappa_a$ is the Prandtl number; κ_a is the thermal conductivity. By retaining only the two variables ξ and p, Eqs. (5) lead to

$$\frac{d\xi}{dy}+\left[1-\left(\frac{c_0k}{\Omega}\right)^2\right]p=-\frac{\rho_w\delta_a^2}{2i\rho_0\Omega}\left[\frac{Q}{T_0}+\frac{k}{\Omega}B_x\right], \tag{7a}$$

$$\frac{dp}{dy}-\left(\frac{\Omega}{c_0}\right)^2\xi=\frac{\delta_a^2}{2}B_y. \tag{7b}$$

The propagation equations, Eqs. (7), must be applied with boundary conditions at the wall. The most appropriate choice to express the boundary conditions would be to use the compliance of the wall which links pressure and normal displacement. But, the liner characteristics are more usually given in terms of the wall admittance. Thus, the boundary condition at the wall is written

$$Y=\frac{\rho_0c_av^*(0)}{p^*(0)}=\frac{v(0)}{p(0)}, \tag{8}$$

where Y is the specific acoustic admittance of the wall.

III. ASYMPTOTIC REPRESENTATION

For simplicity, it is helpful to separate the problem into two regions: (1) the core of the flow, where the dissipation effects can be neglected; (2) a thin layer near the wall, within which the viscous and thermal dissipation effects are confined.

Thus, the problem is amenable to asymptotic analysis. With the method of composite expansions,[11] the solution for any quantity $q(y)$, where $q=u,v,\xi,p,T,\rho$ is expressed as $q=q_c(y)+q_b(\zeta)$ with $\zeta=y/\delta_a$. The second term with subscript b (representing boundary layer terms near the wall $y=0$) tends to zero as $\zeta\rightarrow\infty$. The gradients of mean velocity and temperature are assumed to be non-negligible in the boundary layer. Then, it is convenient to express them, respectively, as $u_a=M_c(y)+M_b(\zeta)$ and $T_o=\Theta_c(y)+\Theta_b(\zeta)$, where the terms with subscript b account for the significant gradients near the wall.

The first terms (inviscid terms in the core of the flow) can be obtained from Eqs. (5) without dissipation and without large gradients near the wall. In the core of the flow, Eqs. (7) lead to the convected wave equation for the outer pressure p_c

$$\frac{d}{dy}\left[\left(\frac{c_0 c}{\Omega_c}\right)^2\frac{dp_c}{dy}\right]+\left[1-\left(\frac{c_{0c}k}{\Omega_c}\right)^2\right]p_c=0, \tag{9}$$

where $\Omega_c=1-kM_c$ and $c_{0c}^2=\Theta_c$. This equation may also be written[10]

$$\frac{d^2p_c}{dy^2}+\left(\frac{1}{\Theta_c}\frac{d\Theta_c}{dy}-\frac{2k}{\Omega_c}\frac{dM_c}{dy}\right)\frac{dp_c}{dy}+\left(\frac{\Omega_c^2}{\Theta_c}-k^2\right)p_c=0, \tag{10}$$

which is the classical Pridmore-Brown[12] equation with temperature gradient. An effective admittance could be defined for the outer region by

$$Y_c=\frac{v_c(0)}{p_c(0)}, \tag{11}$$

while the relationship between the normal velocity and the normal pressure gradient given by Eq. (5c) can be simplified as $i\Omega_c v_c=dp_c/dy$.

In the boundary layer, taking the limit $\delta_a\rightarrow 0$, Eqs. (7) reduce to

$$\frac{d\xi_b}{d\zeta}=-\frac{\rho_w\delta_a}{2i\rho_0\Omega}\left[\frac{Q_b}{T_0}+\frac{k}{\Omega}B_b\right], \tag{12a}$$

$$\frac{dp_b}{d\zeta}=0, \tag{12b}$$

where

$$B_b=\frac{d^2u_b}{d\zeta^2}, \tag{13a}$$

and

$$Q_b=\frac{1}{\sigma^2}\frac{d^2T_b}{d\zeta^2}+2(\gamma-1)\frac{dM_b}{d\zeta}\frac{du_b}{d\zeta}. \tag{13b}$$

The solution of Eq. (12b) (which tends to zero as $\zeta\rightarrow\infty$) is $p_b=0$. Thus, the pressure is constant across the boundary layer to the first order in δ_a.

Integration of Eq. (12a) leads to

$$\xi_b(0)=-\delta_a\int_0^\infty\frac{\rho_w}{2i\rho_{0b}(\zeta)\Omega_b(\zeta)}\times\left[\frac{Q_b}{\Theta_c(0)+\Theta_b(\zeta)}+\frac{kB_b}{\Omega_c(0)+\Omega_b(\zeta)}\right]d\zeta. \tag{14}$$

Without any dissipation, the kinematic condition in the case of a vanishing stationary boundary thickness is continuity of acoustic normal displacement.[9] Using the above notation, this means that $\xi_b(0)=0$. Equation (14) shows that this condition does not hold when there is dissipation, and that an added normal displacement $\xi_b(0)$ is introduced by the viscothermal effects.[13] The expression for the added displacement may be simplified if $\Delta\Theta_0=\Theta_0-\Theta_w$ [where $\Theta_0=\Theta_c(0)$ and $\Theta_w=\Theta_c(0)+\Theta_b(0)$ is the wall temperature] and $M_0=M_c(0)$ are small compared to 1. To the first order in the parameters M_0 and $\Delta\Theta_0$, the added displacement may be written

$$\xi_b(0)=-\frac{i\delta_n}{2}\left(\frac{1}{\Theta_w\sigma^2}\frac{dT_b}{d\zeta}(0)+k\frac{du_b}{d\zeta}(0)\right). \tag{15}$$

The added displacement $\xi_b(0)$, which needs to be included in the boundary conditions, is defined only in terms of the heat flux $q=dT_b/d\zeta$ and the shear stress $\tau=du_b/d\zeta$ at the wall. How these are determined is shown in the next section.

IV. DETERMINATION OF THE ADDED DISPLACEMENT

The diffusion of momentum and heat has to be determined in the boundary layer to find the kinematic condition which can be applied. The momentum equation in the x direction Eq. (5b) becomes, to the first order in M_0 and $\Delta\Theta_0$

$$\frac{d^2u_b}{d\zeta^2}+2iu_b=-2iu_c(0)-\frac{2i}{\delta_a}\frac{dM_b}{d\zeta}\xi_c(0)+2ik\frac{p_c(0)}{\rho_w}. \tag{16}$$

This equation may be transformed into an expression for the shear stress $\tau=du_b/d\zeta$ which is involved in the added displacement

$$\frac{d^2\tau}{d\zeta^2}+2i\tau=\frac{df}{d\zeta}, \tag{17}$$

where

$$f(\zeta)=-\frac{2i}{\delta_a}\frac{dM_b}{d\zeta}\xi_c(0)+2ik\frac{p_c(0)}{\rho_w}. \tag{18}$$

The boundary conditions associated with Eq. (17) are $\tau\rightarrow 0$ when $\zeta\rightarrow\infty$ and $u(0)=u_b(0)+u_c(0)=0$, which can be transformed using Eq. (16) into $d\tau/d\zeta(0)=f(0)$.

The shear waves described by Eq. (17) is excited in two ways. In the first way, a shear wave is excited by the acoustics in the core of flow [i.e., the second term in the definition of $f(\zeta)$]. This wave is the classical one found in dynamic boundary layers near a rigid wall. The second way [which corresponds to the first term in the definition of $f(\zeta)$] is very weak when the wall is rigid [$\xi(0)=0$]. It corresponds to waves induced by changes to the stationary velocity and by the normal displacement near the wall.

Equation (17), subject to the boundary conditions, leads to

$$\tau(0)=-\frac{1}{\tau_1'(0)}\int_0^\infty f(\zeta)\frac{d\tau_1}{d\zeta}d\zeta, \tag{19}$$

where $\tau_1 = \exp[(-1+i)\zeta]$ is the solution of the homogeneous part of Eq. (17) vanishing at infinity. Then, the shear stress at the wall is

$$\tau(0) = (1-i)k\frac{p_c(0)}{\rho_w} + \frac{2i}{\delta_a}M_{\text{eff}}\xi_c(0), \tag{20a}$$

where

$$M_{\text{eff}} = \int_0^\infty \frac{dM_b}{d\zeta}\exp[(-1+i)\zeta]d\zeta \tag{20b}$$

is the effective mean velocity involved in the added displacement. This effective velocity may be seen[14] as an average of the mean velocity over the boundary layer weighted by τ_1 and can be written $M_{\text{eff}} = \beta_v M_0$, where $M_0 = M_c(0)$ with $0 \leqslant |\beta_v| \leqslant 1$.

In the same way, the conservation of energy [Eq. (5d)] leads to a value for the heat flux $q = dT_b/d\zeta$ at the wall

$$\frac{1}{\sigma^2}q(0) = \frac{1-i}{\sigma}(\gamma-1)\frac{p_c(0)}{\rho_w} + \frac{2i}{\delta_a}\Delta\Theta_{\text{eff}}\xi_c(0), \tag{21a}$$

where

$$\Delta\Theta_{\text{eff}} = \int_0^\infty \frac{d\Theta_b}{d\zeta}\exp[(-1+i)\sigma\zeta]d\zeta \tag{21b}$$

is the effective difference between mean temperature and wall temperature involved in the added displacement and can be written $\Delta\Theta_{\text{eff}} = \beta_t\Delta\Theta_0$, where $\Delta\Theta_0 = \Theta_0 - \Theta_w$ with $0 \leqslant |\beta_t| \leqslant 1$.

The added displacement may be written

$$\xi_b(0) = \frac{(1+i)\delta_a}{2\rho_w}\left(\frac{\gamma-1}{\sigma\Theta_w} + k^2\right)p_c(0) - \left(kM_{\text{eff}} + \frac{\Delta\Theta_{\text{eff}}}{\Theta_w}\right)\xi_c(0), \tag{22}$$

which leads to a relation giving the modified admittance

$$\left(1-(1-\beta_v)kM_0 + \frac{\Delta\Theta_0}{\Theta_w}\beta_t\right)Y_c = Y + \frac{1-i}{2\rho_w}\left(\frac{\gamma-1}{\sigma} + k^2\right)\delta_a. \tag{23}$$

It can be seen from Eq. (23) that the effect of the classical shear and thermal waves (induced by the acoustics in the core of the flow) which leads to the second term on the right-hand side of (23) is weak (of the order δ_a). Since Y is much larger than δ_a for a typical lined wall, this term is only important for a hard wall and will be neglected in front of Y in what follows.

When β_v and $\beta_t \to 0$, Eq. (23) is equivalent to the continuity of acoustic normal displacement across the boundary layer: $\xi_b(0) = 0$ or $Y_c = Y/(1-kM_0)$. When β_v and $\beta_t \to 1$, Eq. (23) is transformed in a condition of conservation of the normal mass velocity across the boundary layer: $\rho_c(0)v_c(0) = \rho_w v(0)$ or $\Theta_0 Y_c = \Theta_w Y$.

This behavior is illustrated here for three simplified mean velocity profiles with a constant temperature. The outer mean velocity is taken as constant, i.e., $M_c(y) = M_0$. The slope at the origin is the same for all three profiles: $du_0/dy(0) = M_0/\Delta$ where Δ is the stationary boundary layer thickness [see Fig. 2(b)]. The ratio of the acoustic over the stationary boundary layer thickness is called $\delta^* = \delta_a/\Delta$.

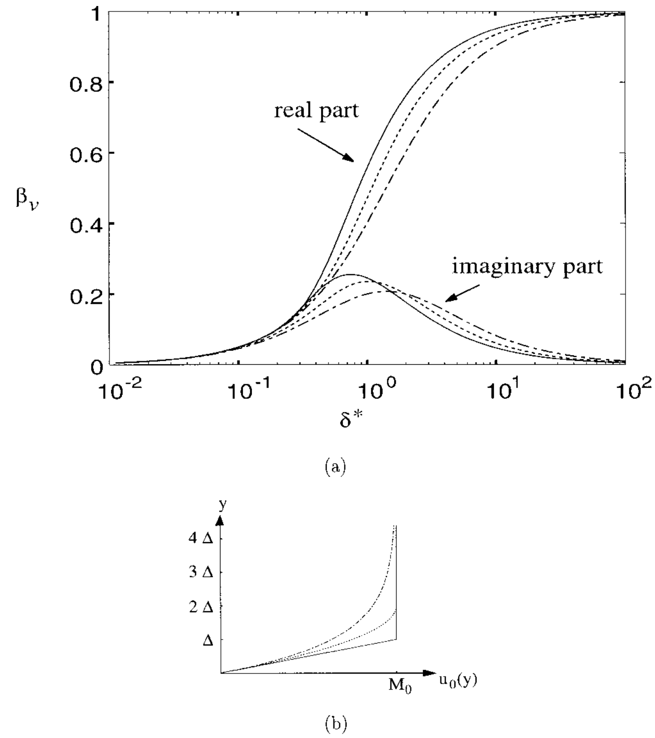

FIG. 2. (a) Variation of the effective velocity divided by the core velocity, β_v, as a function of the ratio of the acoustic and stationary boundary layer thicknesses, δ^*, for three mean velocity profiles. Solid line: linear; dashed line: quadratic, and dash-dot line: exponential. (b) Mean velocity profiles.

(1) For the first profile, the inner mean velocity is linear

$$M_b(\zeta) = -M_0(1-\delta^*\zeta) \quad \text{for } 0 \leqslant \zeta \leqslant 1/\delta^*,$$

$$M_b(\zeta) = 0 \quad \text{for } \zeta > 1/\delta^*,$$

and in this case

$$\beta_v = \frac{(1+i)\delta^*}{2}\left[1-\exp\left(\frac{-1+i}{\delta^*}\right)\right]. \tag{24}$$

(2) The second profile is quadratic

$$M_b(\zeta) = -M_0(1-\delta^*\zeta/2)^2 \quad \text{for } 0 \leqslant \zeta \leqslant 2/\delta^*,$$

$$M_b(\zeta) = 0 \quad \text{for } \zeta > 2/\delta^*,$$

and in this case

$$\beta_v = \frac{i\delta^*}{2}\left[\left(\exp\left(\frac{2(-1+i)}{\delta^*}\right)-1\right)\frac{\delta^*}{2} - (-1+i)\right]. \tag{25}$$

(3) The last profile is exponential, i.e., $M_b(\zeta) = -M_0\exp(-\delta^*\zeta)$, and in this case $\beta_v = \delta^*/(\delta^*+1-i)$.

The real and imaginary parts of β_v as a function of δ^* are plotted in Fig. 2(a) for the three profiles. When the acoustic boundary layer thickness, δ_a, is small compared to the stationary boundary layer thickness Δ (i.e., $\delta^* \ll 1$), β_v goes to zero. In this case, continuity of displacement can be applied across the boundary layer. On the other hand, when $\delta^* \gg 1$, β_v goes to 1, which means that continuity of velocity is applicable across the boundary layer. For a given station-

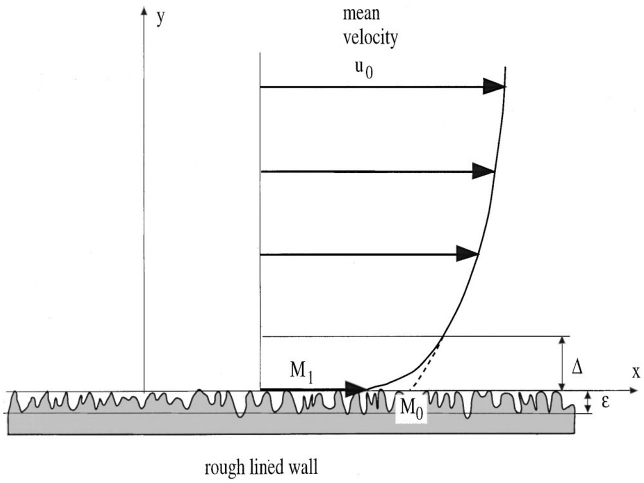

FIG. 3. Schematic description of the rough wall geometry.

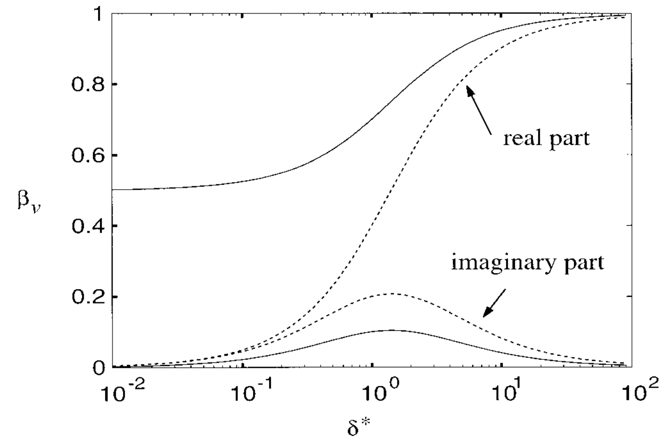

FIG. 4. Variation of the effective velocity divided by the core velocity, β_v, as a function of the ratio of the acoustic and stationary boundary layer thickness, δ^*, for a rough wall with an exponential mean velocity profile. Solid line: rough wall; dashed line: smooth wall [same as Fig. 2(a)].

ary boundary layer thickness, Δ, continuity of displacement applies at high frequencies while continuity of mass velocity applies at low frequencies. For a given frequency, continuity of displacement applies at low Mach number (i.e., giving a thick stationary boundary layer), while continuity of mass velocity applies at high Mach number (i.e., resulting in a thin stationary boundary layer). These findings are in qualitative agreement with the experimental observations of Ingard and Singhal.[15]

It should be noted that, when both acoustic and stationary boundary layer thicknesses are of the same order $\Delta \simeq \delta_a$, β_v and β_t are complex, so they not only change the value but also the character of the admittance.

It may be seen from Eq. (20a) that the most important part of the acoustic shear stress comes from the transfer, by the normal fluctuating displacement, of axial momentum from the stationary flow into the lined wall,[16] this effect being induced by viscosity. The parameter β_v controls this transfer from $\beta_v = 0$ (no transfer) to $\beta_v = 1$ (full transfer). β_v can be seen from Eq. (20b) to be the ratio of the mean velocity in the layer where the shear wave is significant over the core mean velocity. The same reasoning holds for the thermal flux and the parameter β_t.[14]

V. ADMITTANCE OF A ROUGH LINED WALL

The mean velocity profiles of turbulent flow over a rough wall is schematically depicted in Fig. 3. Compared to a smooth wall profile, the main difference is the slip velocity M_1 at the outer boundary of the equivalent roughness thickness. This slip velocity depends on the equivalent roughness and on the core velocity.[17]

Taking into account the above analysis of the viscous effects, the axial momentum linked to the slipping velocity M_1 is transferred into the roughness of the wall even if the acoustic boundary layer thickness is small compared to Δ. Then, the continuity of mass velocity must be applied over a distance equal to the roughness of the wall.[13] This can be written as $\rho_1 v(0) = -i(1-kM_1)\rho_1 \xi(0) = \rho_w Y p(0)$, where ρ_1 and ρ_w are the density corresponding, respectively, to the $y=0$ and to the wall temperature. The origin of the coordinates y and ζ is taken at the outer boundary of the equivalent roughness thickness ϵ. Assuming that the axial acoustic velocity is equal to 0 at $y=0$, Eq. (23) then becomes

$$\left(1-(1-\tilde{\beta}_v)kM_0+\frac{\Delta\Theta_a}{\Theta_w}\tilde{\beta}_t\right)Y_c = Y, \tag{26}$$

where

$$\tilde{\beta}_v = \frac{1}{M_0}\left(M_1+\int_0^\infty \frac{dM_b}{d\zeta}\exp[(-1+i)\zeta]d\zeta\right), \tag{27a}$$

and

$$\tilde{\beta}_t = \frac{1}{\Delta\Theta_0}\left(\Delta\Theta_1+\int_0^\infty \frac{d\Theta_b}{d\zeta}\exp[(-1+i)\sigma\zeta]d\zeta\right); \tag{27b}$$

$\Delta\Theta_1 = \Theta_w - \Theta_1$ is the difference between the wall temperature and the temperature at $y=0$.

The effect of roughness is illustrated for the case of an exponential velocity profile with a slip velocity, in the case of a constant temperature. The stationary velocity profile takes the form $M_b(\zeta) = -(M_0-M_1)\exp(-\delta^*\zeta)$ and in this case $\beta_v = (\delta^* + (1-i)M_1/M_0)/(\delta^*+1-i)$.

The real and imaginary parts of $\tilde{\beta}_v$ as a function of δ^* are plotted in Fig. 4 for $M_1 = 0.5M_0$. It can be seen that continuity of normal displacement ($\tilde{\beta}_v = 0$) is never attained for a rough wall (solid line in Fig. 4). When the acoustic boundary layer thickness δ_a is small compared to the stationary boundary layer thickness Δ (i.e., $\delta^* \ll 1$), $\beta_v \to M_1/M_0$, and the boundary condition is $Y_c = Y/(1-kM_1)$ instead of $Y_c = Y/(1-kM_0)$ for the case of a smooth wall (i.e., continuity of displacement).

VI. CONCLUSIONS

The effective acoustic admittance of a liner, taking account of viscothermal effects, can be computed for the case where acoustic and stationary boundary layer thicknesses are small compared to the wavelength. The main effect of viscosity is the transfer of axial momentum and heat flux of the stationary flow into the lined wall. The effective admittance is given as a function of two coefficients β_v and β_t which mainly depend on the ratio of the acoustic and stationary boundary layer thicknesses. When the acoustic boundary layer thickness is small compared with the stationary boundary layer thickness, continuity of normal displacement applies across the boundary layer. On the other hand, when the acoustic boundary layer thickness is large compared with the stationary boundary layer thickness, it is continuity of mass velocity which applies across the boundary layer. If the lined wall is rough, normal displacement continuity never applies. In this paper, only molecular diffusion effects (described by the dynamic viscosity and the thermal conductivity) are taken into account. Further work is needed to include the turbulent diffusion effects which can be incorporated in a complex effective viscosity and which will depend both on the normal coordinate and on frequency.

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